

PARTIALLY HYPERBOLIC SETS WITH A DYNAMICALLY MINIMAL INVARIANT LAMINATION.

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ABSTRACT. We study partially hyperbolic sets of C^1 -diffeomorphisms. For these sets there are defined the strong stable and strong unstable laminations. A lamination is called dynamically minimal when the orbit of each leaf intersects the set densely.

We prove that partially hyperbolic sets having a dynamically minimal lamination have empty interior. We also study the Lebesgue measure and the spectral decomposition of these sets. These results can be applied to C^1 -generic/robustly transitive attractors with one-dimensional center bundle.

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1. INTRODUCTION

Hyperbolicity of a proper set imposes quite specific properties of its “size” and “structure”, especially when the dynamics on it is transitive. For instance, it is well known that transitive hyperbolic proper sets have empty interior. This is proved using the saturation principle in [12]¹. Bowen proved in [11] that C^2 hyperbolic horseshoes have zero Lebesgue measure. The proof of this result involves bounded distortion arguments as well as the absolute continuity of the foliations, ingredients which are not available for maps with less regularity. Indeed, [10] provided an example of C^1 hyperbolic horseshoe with positive Lebesgue measure.

Similar results were obtained for non-hyperbolic dynamics assuming a weaker form of hyperbolicity known as partial hyperbolicity. A set $\Lambda \subset M$ is *partially hyperbolic* for a diffeomorphism $f : M \rightarrow M$ if the tangent bundle $T_\Lambda M$ over the set Λ has a dominated splitting into three Df -invariant subbundles $E^s \oplus E^c \oplus E^u$, where E^s and E^u are uniformly expanded by Df and Df^{-1} , respectively. When E^s , E^c , and E^u are all nontrivial, we speak of *strongly partially hyperbolic* sets.

The results in [2] study the case when the non-wandering set $\Omega(f)$ is partially hyperbolic and has non-empty interior. Recall that C^1 -generically² the set $\Omega(f)$ splits into pairwise disjoint homoclinic classes³ which are its elementary pieces and form its spectral decomposition, see [5] and Definition 4.1. It is proved that a strongly partially hyperbolic homoclinic class with non-empty interior is the whole manifold. Moreover, when the whole manifold is partially hyperbolic, this result holds C^1 -openly. Similar results were obtained in [18] assuming that the homoclinic class is *bi-Lyapunov stable*, which is a slightly more general condition than having non-empty interior.

Finally, considering again the Lebesgue measure of invariant transitive sets and in the same spirit of [11], the results in [4] extended Bowen’s result to the partially hyperbolic setting by showing that sufficiently regular diffeomorphisms (of a class of differentiability bigger than one) have no “horseshoe-like” partially hyperbolic sets with positive Lebesgue measure.

In this work we deal with partially hyperbolic transitive sets Λ of C^1 -diffeomorphisms. We provide sufficient conditions guaranteeing that these sets have empty interior or zero Lebesgue measure. A key feature in this setting is the existence of invariant dynamically defined *laminations* integrating the bundles E^s and E^u , that we denote by \mathcal{F}^s and \mathcal{F}^u , respectively. When for each leaf of the lamination its orbit has a dense intersection with Λ ,

¹By *saturation* we mean the saturation of a set by the leaves of the stable and unstable foliations, see also Definition 2.1. For non-transitive sets, in [14] there is an example of a hyperbolic proper set with robustly non-empty interior.

²We say that a property holds C^1 -generically if it holds for a residual (G_δ and C^1 -dense) subset of the space of C^1 diffeomorphisms.

³A homoclinic class is a (not necessarily hyperbolic) generalisation of a horseshoe: it is a transitive set associated to a hyperbolic periodic point p defined as the closure of the transverse intersections of the stable and unstable manifolds of p .

the lamination is said to be *dynamically minimal* (see Definition 3.1). In this case, we say that Λ is an *s-minimal* or *u-minimal set*, according to which lamination (\mathcal{F}^s or \mathcal{F}^u) is dynamically minimal. In [16] we prove that there is a wide class of systems verifying this property: robustly/generically transitive attractors with one-dimensional center bundle (see also [6, 15] for previous results in this direction).

Our main result (Theorem A) claims that *u*- and *s*-minimal proper sets have empty interior. Assuming that the central bundle is one-dimensional we prove that, C^1 -generically, *s*-minimal proper attractors have zero Lebesgue measure (see Theorem C).

Another motivation of this paper concerns the spectral decomposition results for sets containing the relevant part of the dynamics (limit, non-wandering, chain-recurrent sets, etc.). In the classical hyperbolic case, this decomposition consists of finitely many sets, called *basic pieces*, which each is a homoclinic class, see [19]. Specially important sets in this decomposition are the attractors and the repellers, which are persistent and robustly transitive and whose basins form an open and dense subset of the ambient space. There are some non-hyperbolic counterparts for this decomposition based on Conley's theory (see [5, 13]). More recently, [3] states a C^1 -generic spectral decomposition theorem for chain-transitive locally maximal sets. Here we prove a spectral decomposition theorem for *s*- and *u*-minimal homoclinic classes, see Theorems D and E.

1.1. Statement of the results. The precise definitions and notations involved in the results in this section can be found in Section 2.

Theorem A. *Every s- or u-minimal proper set has empty interior.*

From Theorem B in [16] (see also item (2) of Proposition 2.5 in this paper), we get immediately the following corollary.

Corollary B. *A C^1 -generic robustly transitive partially hyperbolic proper attractor with one-dimensional center bundle has robustly empty interior.*

In the next statement, $\Lambda_f(U)$ denotes the maximal invariant set of f in the open set U .

Theorem C. *For a generic $f \in \text{Diff}^1(M)$, let $\Lambda_f(U)$ be a partially hyperbolic s-minimal proper attractor with one-dimensional center bundle. Then there are a neighborhood \mathcal{U} of f , an open and dense subset $\mathcal{V} \subset \mathcal{U}$, and a residual subset \mathcal{W} of \mathcal{U} such that:*

- (1) $\Lambda_g(U)$ has empty interior for all $g \in \mathcal{V}$.
- (2) $\Lambda_g(U)$ has zero Lebesgue measure for all $g \in \mathcal{W}$.

Moreover, the set \mathcal{W} contains every $C^{1+\alpha}$ diffeomorphism in \mathcal{V} , for every $\alpha > 0$.

Observe that item (1) of Theorem C is stronger than Corollary B, as we get robustly empty interior even if the attractor is not robustly transitive. Unfortunately, this is only obtained for the *s*-minimal case.

Finally, we state a spectral decomposition theorem for s - and u -minimal homoclinic classes. Here the term *minimal constant* stands for the smallest number d verifying the definition of a dynamically minimal lamination (see Definition 3.1). We denote by $H(p, f)$ the homoclinic class of the hyperbolic periodic point p and by $\text{index}(p)$ the dimension of the stable manifold of p .

Theorem D. *Let $\Lambda = H(p, f)$ be an s -minimal (resp. u -minimal) isolated partially hyperbolic homoclinic class with minimal constant d and $\text{index}(p) = \dim(E^s)$ (resp. $\text{index}(p) = \dim(E^s) + \dim(E^c)$). Then Λ admits a unique spectral decomposition with exactly d components.*

As a consequence of Theorem B in [16], we obtain a robust spectral decomposition for robustly transitive attractors, meaning that every g in a small neighborhood of f has a spectral decomposition whose pieces are the continuations of the pieces in the spectral decomposition of Λ_f .

Theorem E. *There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ satisfying the following. For every $f \in \mathcal{R}$ and $U \subset M$, if $\Lambda_f(U)$ is a partially hyperbolic robustly transitive attractor with one-dimensional center bundle, then $\Lambda_f(U)$ has a robust spectral decomposition.*

This paper is organized as follows. In Section 2 we give the basic definitions, terminology, and state some results we use along the paper. Theorem A is proved in subsection 3.1, Theorem C is proved in section 3.2, and Theorems D and E are proved in section 4.

2. PRELIMINARIES

Let M be a Riemannian compact manifold without boundary and, for $r \geq 1$, let $\text{Diff}^r(M)$ be the space of C^r diffeomorphisms from M to itself endowed with the C^r -topology.

Given $f \in \text{Diff}^1(M)$ and an open subset U of M , we define the maximal f -invariant set of f in U by

$$\Lambda_f(U) := \bigcap_{n \in \mathbb{Z}} f^n(U).$$

When a compact set Λ is the maximal f -invariant set of some open set $U \subset M$, we say that Λ is an *isolated* set. Isolated sets vary upper semicontinuously. By an abuse of terminology, we call the set $\Lambda_g(U)$ the *continuation* of the set $\Lambda_f(U)$ when g varies in a small neighborhood of f .

A special kind of isolated set are *attractors*. We say that a set Λ is an attractor if there is an open set $U \subset M$ such that $\Lambda = \bigcap_{n \in \mathbb{N}} f^n(U)$ and $f(\overline{U}) \subset U$. Observe that M itself is an attractor (by taking $U = M$). The interesting case is when $\Lambda \neq M$, when Λ is called a *proper* attractor.

In this work we study isolated sets with highly recurrent dynamics. We say that a set Λ is *transitive* if there is $x \in \Lambda$ such that its forward orbit $\mathcal{O}_f^+(x)$ is dense in Λ . In our setting, this is equivalent to the following property:

For any pair V_1, V_2 of (relative) non-empty open sets of Λ , there is $n \in \mathbb{Z}$ such that $f^n(V_1) \cap V_2 \neq \emptyset$. A stronger recurrence property is the *mixing* property: For any pair V_1, V_2 of (relative) open sets of Λ , there is $n \in \mathbb{N}$ such that $f^m(V_1) \cap V_2 \neq \emptyset$ for all $m \geq n$.

We speak of a *robustly* transitive set $\Lambda = \Lambda_f(U)$ when Λ is transitive and the transitivity is also verified for the continuations $\Lambda_g(U)$ of every g in a small neighborhood \mathcal{U} of f . If the transitivity is verified only in a residual subset of \mathcal{U} , then we say that Λ is a *generically transitive set*.

In our context the isolated sets Λ are always assumed to be partially hyperbolic with $E^s \oplus E^c \oplus E^u$ denoting the partially hyperbolic splitting of $T_\Lambda M$. The values of $\dim(E^s)$, $\dim(E^c)$ and $\dim(E^u)$, are designated by d^s , d^c , and d^u , respectively. We also assume that Λ is robustly non-hyperbolic, meaning that E^c does not have uniform contraction nor expansion in a robust way. We also require that none of the three bundles are trivial, in which case the set is strongly partially hyperbolic. See Appendix B of [7] for a list of elementary properties and a more complete view on this topic.

Partial hyperbolicity leads to the existence of dynamically defined immersed submanifolds $\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$, through each point x in the set, tangent to the stable and unstable subbundles, respectively. The set of such submanifolds are known as the stable and unstable *lamination* of the set and are denoted by \mathcal{F}^s and \mathcal{F}^u , respectively. We direct the reader to section 3 of [16], where the precise definition and main properties of these laminations are provided.

When dealing with perturbations of a diffeomorphism, as in the case of the continuations of isolated sets, we need to specify in these notations which diffeomorphism we are referring to. So, let $\Lambda_f(U)$ be an isolated partially hyperbolic set and \mathcal{U} be a neighborhood of f such that, for every $g \in \mathcal{U}$, the set $\Lambda_g(U)$ is partially hyperbolic with the same bundles dimensions. We denote by $\mathcal{F}^s(g)$ and by $\mathcal{F}^s(x, g)$, respectively, the strong stable lamination of $\Lambda_g(U)$ (with respect to the partial hyperbolicity of g) and the leaf of this foliation that contains x . Similarly, given a hyperbolic periodic point $x \in \Lambda_g(U)$ and $\varepsilon > 0$, we denote by $W_\varepsilon^s(x, g)$ and $W^s(x, g)$ the local stable manifold (of size ε) and the global stable manifolds of x , respectively. The union of all local or all global stable manifolds along the orbit of x is denoted by $W_\varepsilon^s(\mathcal{O}_g(x), g)$ and $W^s(\mathcal{O}_g(x), g)$, respectively. Similarly, fixed $r > 0$, we denote by $\mathcal{F}_r^s(x)$ the open ball of radius r centered at x , relative to the induced distance on $\mathcal{F}^s(x)$. When there is no risk of misunderstanding, we simplify these notation by omitting the diffeomorphism, as $\mathcal{F}^s(x)$ for $\mathcal{F}^s(x, f)$, $W^s(x)$ for $W^s(x, f)$, and $W_\varepsilon^s(\mathcal{O}_g(x))$ for $W_\varepsilon^s(\mathcal{O}_g(x), g)$.

Similar notations are considered for the unstable foliation and manifold.

Definition 2.1. The *saturation* of a set K by a lamination \mathcal{F} is the set consisting of the union of all the leaves passing through some point of K . A

set K is *saturated* by \mathcal{F} if the saturation of K equals K (i.e, for every $x \in K$ we have $\mathcal{F}(x) \subset K$).

Remark 2.2. Let Λ be a partially hyperbolic set. For every hyperbolic periodic point $p \in \Lambda$, the *index* of p is the dimension of $W^s(p)$ as a submanifold and is denoted by $\text{index}(p)$. Since $\mathcal{F}^s(p)$ is a subset of $W^s(p)$, we have $\text{index}(p) \geq d^s$. Analogously, the strong unstable leaf of p is a subset of $W^u(p)$ so $d^s + d^c + d^u - \text{index}(p) \geq d^u$. In particular, when the central bundle is one-dimensional ($d^c = 1$), the index of a hyperbolic periodic point p is either d^s or $d^s + 1$.

Following [16], given a diffeomorphism f and an isolated set $\Lambda = \Lambda_f(U)$, we define the concept of *compatible neighbourhood* of f , where the continuations of $\Lambda_f(U)$ share its main properties.

Definition 2.3. Let Λ be an isolated set of a diffeomorphism $f \in \text{Diff}^1(M)$ and $U \subset M$ an isolated block of Λ . We call a neighborhood \mathcal{U} of f a *compatible* neighborhood (with respect to U) if \mathcal{U} is sufficiently small so that, for all $g \in \mathcal{U}$:

- the set $\Lambda_g(U)$ is isolated;
- if $\Lambda_f(U)$ is an attractor of f , then $\Lambda_g(U)$ is an attractor of g ;
- if $\Lambda_f(U)$ is a partially hyperbolic set then $\Lambda_g(U)$ is a partially hyperbolic set of g , with the same bundles dimensions;
- if $\Lambda_f(U)$ is a generically (resp. robustly) transitive set of f , then $\Lambda_g(U)$ is a generically (resp. robustly) transitive set of g .

2.1. Generic Isolated Sets and Attractors.

In this section we gather some useful results that we invoke along our proofs. They were established in [1, 5, 8, 16, 17]. For convenience, we restate them here in a compact form.

Proposition 2.4. *There is a residual subset \mathcal{R} of $\text{Diff}^1(M)$ such that, for every $f \in \mathcal{R}$ and every isolated set $\Lambda_f(U)$, it holds:*

- (1) *if $\Lambda_f(U)$ is a transitive attractor, then there is a neighborhood \mathcal{U} of f such that, for every $g \in \mathcal{R} \cap \mathcal{U}$, the set $\Lambda_g(U)$ is a transitive attractor.*
- (2) *if $\Lambda_f(U)$ is non-hyperbolic, then it contains a pair of (hyperbolic) saddles of different indices.*
- (3) *if $\Lambda_f(U)$ is a transitive isolated set of f that is partially hyperbolic with one-dimensional center bundle, then for every pair of hyperbolic periodic points $p, q \in \Lambda_f(U)$ with indices d^s and $d^s + 1$, respectively, there is an open set $\mathcal{V}_{p,q} \subset \text{Diff}^1(M)$, with $f \in \overline{\mathcal{V}_{p,q}}$, satisfying:
 $W^s(\mathcal{O}_g(q_g)) \subset \overline{W^s(\mathcal{O}_g(p_g))}$ and $W^u(\mathcal{O}_g(p_g)) \subset \overline{W^u(\mathcal{O}_g(q_g))}$ for every $g \in \mathcal{V}_{p,q}$. Moreover, if $\Lambda_f(U)$ is robustly transitive, then $\Lambda_g(U) \subset H(p_g, q_g)$.*

- (4) if $\Gamma = H(p, f)$ is a partially hyperbolic homoclinic class, then there is an extension of the partially hyperbolic splitting on Γ to a continuous splitting on a compact neighborhood W of Γ such that it is invariant in the following sense: for every $x \in W$ with $f(x) \in W$, we have that $Df_x(E^i(x)) = E^i(f(x))$, for any $i \in \{s, c, u\}$.
- (5) if $\Lambda_f(U)$ is an s -minimal partially hyperbolic set with one-dimensional center bundle and \mathcal{U} is a compatible neighborhood of f , then for every hyperbolic periodic point $p \in \Lambda_f(U)$, there is an open set $\mathcal{W}_p \subset \mathcal{U}$, with $f \in \overline{\mathcal{W}_p}$, such that $H(p_g, g) \subset \overline{\mathcal{O}_g^-(D)}$ for every strong stable disk D centered at some point $x \in \Lambda_g(U)$ and every $g \in \mathcal{W}_p$. Moreover, if $\text{index}(p) = d^s$, then \mathcal{W}_p is a neighborhood of f .

Item (1) is theorem B of [1]; item (2) is due to Mane in the proof of the Ergodic Closing Lemma [17]; item (3) is Proposition 4.8 in [16]; item (4) is Theorem 5.1 in [16] (which is a combination of Theorem 7 in [8] and Remark 1.10 in [5]); and item (5) is Lemma 9.4 in [16].

In the rest of this paper, \mathcal{R} always refers to the residual subset in Proposition 2.4.

Fixed an open set $U \subset M$, denote by $\text{RTPHA}_1(U)$ (resp. $\text{GTPHA}_1(U)$) the subset of $\text{Diff}^1(M)$ of diffeomorphisms f for which the maximal f -invariant subset $\Lambda_f(U)$ of U is a robustly (resp. generically) transitive attractor that is robustly non-hyperbolic and partially hyperbolic with one-dimensional center bundle. Observe that $\text{RTPHA}_1(U)$ is an open subset of $\text{Diff}^1(M)$, and that $\text{GTPHA}_1(U)$ is locally residual in $\text{Diff}^1(M)$.

Next proposition summarises Theorem A, Theorem B, and Corollary 4.9 in [16].

Proposition 2.5 ([16]). *For every open subset $U \subset M$, there is a residual subset \mathcal{A} of $\text{GTPHA}_1(U)$ and an open and dense subset \mathcal{B} of $\text{RTPHA}_1(U)$ such that:*

- (1) *for every $g \in \mathcal{A}$, the set $\Lambda_g(U)$ is either generically s -minimal or generically u -minimal.*
- (2) *for every $g \in \mathcal{B}$, the attractor $\Lambda_g(U)$ is either robustly s -minimal or robustly u -minimal. Moreover, $\Lambda_g(U)$ is a homoclinic class and depends continuously on $g \in \mathcal{B}$.*

2.2. Lebesgue Measure and Genericity.

In what follows we consider the manifold M endowed with a Lebesgue measure m . We see how Lebesgue measure behaves for the perturbations of an isolated set. Observe that every isolated set $\Lambda_f(U)$ is m -measurable, as it is a countable intersection of open sets.

Lemma 2.6. *Let f be a diffeomorphism in $\text{Diff}^1(M)$, $\Lambda_f(U)$ be an isolated set, and \mathcal{U} be a compatible neighborhood of f with respect to $\Lambda_f(U)$. The map $\varphi: \mathcal{U} \rightarrow \mathbb{R}$ defined by $\varphi(g) = m(\Lambda_g(U))$ is upper semicontinuous. Consequently, the set of continuity points of the map φ is a residual subset of \mathcal{U} .*

Proof. Fix $g \in \mathcal{U}$ and consider the nested sequence of open sets $\Lambda(g, k) := \bigcap_{n=-k}^k g^n(U)$. Clearly, $\Lambda(g, k) \searrow \Lambda_g(U)$ as $k \rightarrow \infty$. Since m is a regular measure, we obtain $\lim_{k \rightarrow \infty} m(\Lambda(g, k)) = m(\Lambda_g(U))$. Thus, fixed $\varepsilon > 0$, there is $N = N(g, \varepsilon) \in \mathbb{N}$ such that

$$m(\Lambda(g, k)) < m(\Lambda_g(U)) + \varepsilon = \varphi(g) + \varepsilon, \quad \text{for all } k \geq N.$$

Note that there is $N_0 \in \mathbb{N}$ such that the closure of $\Lambda(g, N + N_0)$ is contained in the open set $\Lambda(g, N)$. Then, for every h sufficiently close to g , it holds that $\Lambda(h, N + N_0) \subset \Lambda(g, N)$. Hence,

$$m(\Lambda_h(U)) \leq m(\Lambda(h, N + N_0)) \leq m(\Lambda(g, N)) \leq m(\Lambda_g(U)) + \varepsilon.$$

This means that $\varphi(h) \leq \varphi(g) + \varepsilon$, implying the lemma. \square

By an standard result of topology, we get the following consequence.

Corollary 2.7. *Under the hypotheses and with the notation of Lemma 2.6, if there is a dense subset \mathcal{W} of \mathcal{U} such that $\varphi(g) = 0$ for all $g \in \mathcal{W}$, then there is a residual subset \mathcal{G} of \mathcal{U} such that $\varphi(g) = 0$ for all $g \in \mathcal{G}$.*

Remark 2.8. Lemma 2.6 and Corollary 2.7 hold for attractors, as any attractor is an isolated set.

3. DYNAMICALLY MINIMAL LAMINATIONS

3.1. u - and s -minimal sets.

For notational simplicity, given a strongly partially hyperbolic set Λ we adopt the following notation.

$$\mathcal{F}_\Lambda^s(x) = \mathcal{F}^s(x) \cap \Lambda \quad \text{and} \quad \mathcal{F}_\Lambda^u(x) = \mathcal{F}^u(x) \cap \Lambda.$$

Definition 3.1 (dynamically minimal lamination). Let Λ be a partially hyperbolic set of a diffeomorphism f with nontrivial stable bundle E^s . We say that the lamination \mathcal{F}^s is *dynamically minimal* (or Λ is an *s-minimal set*) if there is $d \in \mathbb{N}$ such that, for all $x \in \Lambda$, it holds that

$$\bigcup_{i=1}^d \overline{\mathcal{F}_\Lambda^s(f^i(x))} = \Lambda.$$

When $\Lambda = \Lambda_f(U)$ is an isolated set, Λ is a *robustly s-minimal set* if $\Lambda_g(U)$ is *s-minimal* for all g in a neighborhood \mathcal{U} of f . If *s-minimality* is verified only in a residual subset of \mathcal{U} , then $\Lambda_f(U)$ is called a *generically s-minimal set*.

The definition of u -minimality is analogous, considering the strong unstable lamination \mathcal{F}^u .

The smallest natural number d verifying this definition is called the *minimal constant* of Λ . The reason we need such number d of iterates to obtain the desired density property is that the attractor may not be a unique elementary piece. In fact, we prove in Section 4 that the minimal constant d is exactly the number of pieces in the spectral decomposition of Λ (see Definition 4.1). Moreover, when $\Lambda = M$, then $d = 1$, so the definition of u - and s -minimality coincides with the definition of minimal foliation for partially hyperbolic diffeomorphisms.

The main result in this section is the following equivalence of Theorem A.

Theorem 3.2. *Any u - or s -minimal set with non-empty interior is the whole manifold.*

In the rest of this section, all the results are stated for s -minimal sets, though similar statements (with similar proofs) also hold in the u -minimal case.

We start with some auxiliary lemmas and the following Remark, that gives two well known properties of the strong stable.

Remark 3.3. For every $r > 0$ sufficiently small, it hold:

- i) $\mathcal{F}^s(x) = \bigcup_{n \in \mathbb{N}} f^{-n}(\mathcal{F}_r^s(f^n(x)))$
- ii) There is $N \in \mathbb{N}$ such that $A_n(x) = f^{-n.N}(\mathcal{F}_r^s(f^{n.N}(x)))$ yield a nested sequence (that is, $A_n(x) \subset A_{n+1}(x)$ for every $n \in \mathbb{N}$).

Given a set $K \subset M$, we denote by $B_\varepsilon(K)$ the ε -neighborhood of K relative to some fixed Riemannian metric on M .

Lemma 3.4. *Let Λ be an s -minimal set of a diffeomorphism f and d be its minimal constant. Given any $\varepsilon > 0$ and $r > 0$ sufficiently small, there is a constant $N = N(\varepsilon, r) \in \mathbb{N}$ such that*

$$\Lambda \subset B_\varepsilon \left(\bigcup_{i=1}^d f^{-k.N+i}(\mathcal{F}_r^s(x)) \right) \quad \text{for all } x \in \Lambda \text{ and } k \in \mathbb{N}.$$

Proof. Fix $\varepsilon > 0$ and $r > 0$. From s -minimality and Remark 3.3, given any $y \in \Lambda$, there is $N_y \in \mathbb{N}$ such that

$$\Lambda \subset B_\varepsilon \left(\bigcup_{i=1}^d f^i(f^{-N_y}(\mathcal{F}_r^s(f^{N_y}(y)))) \right).$$

By the continuity of the foliation \mathcal{F}^s , there is a neighborhood $V(y)$ of y such that the previous inclusion holds for all $z \in V(y) \cap \Lambda$, with $N_z = N_y$.

Consider the covering $\{V(y)\}_{y \in \Lambda}$ of Λ . Since Λ is a compact set, we may extract a finite subcovering $\{V(y_i)\}_{i=1}^m$ and constants N_{y_i} such that, if $y \in \Lambda \cap V(y_j)$ for some $j \in \{1, \dots, m\}$, then

$$\Lambda \subset B_\varepsilon \left(\bigcup_{i=1}^d f^i(f^{-N_j}(\mathcal{F}_r^s(f^{N_j}(y)))) \right).$$

Let $N = \text{LCM}(N_1, N_2, \dots, N_m)$ be the least common multiple of these numbers. By item *ii*) of Remark 3.3, we can replace N_j by any natural number $k.N$, with $k \in \mathbb{N}$, so we have

$$\Lambda \subset B_\varepsilon \left(\bigcup_{i=1}^d f^i(f^{-k.N}(\mathcal{F}_r^s(f^{k.N}(y)))) \right), \quad \text{for every } y \in \Lambda \text{ and } k \in \mathbb{N}.$$

Given $x \in \Lambda$ and $k \in \mathbb{N}$ we set $y = f^{-k.N}(x)$ in the above inclusion, so we obtain the lemma. \square

Lemma 3.5. *Let Λ be an s -minimal set of a diffeomorphism f . If Λ contains some strong stable disk, then Λ contains the strong stable leaf of every point in Λ .*

Proof. Let $r > 0$ and $x_0 \in \Lambda$ be such that the strong stable disk $D = \mathcal{F}_r^s(x_0)$ is contained in Λ , and let $y \in \Lambda$ be an accumulation point of the backward orbit of x_0 .

Fix $\delta > 0$ sufficiently small so that, by the partial hyperbolicity on Λ , there is $m_0 \in \mathbb{N}$ such that for every stable disk \mathcal{S} of length δ and $m \geq m_0$, the image $f^m(\mathcal{S})$ is contained inside a stable disk of radius r . Hence, there is an increasing sequence $\{n_i\}_{i \in \mathbb{N}} \subset \mathbb{N}$, with $n_i \geq m_0$, such that $\lim_{i \rightarrow \infty} f^{-n_i}(x_0) = y$ and, for every $i \in \mathbb{N}$, the disk $f^{-n_i}(D)$ has inner radius bigger than δ . By the continuity of the lamination, we obtain that $\mathcal{F}_\delta^s(y) \subset \Lambda$. For every $m \in \mathbb{N}$, the point $f^{-m}(y)$ is also an accumulation point of the backward orbit of x_0 , so the same argument leads to $\mathcal{F}_\delta^s(f^m(y)) \subset \Lambda$. Then we conclude that $f^{-m}(\mathcal{F}_\delta^s(f^m(y))) \subset \Lambda$ for every $m \in \mathbb{N}$, which implies that $\mathcal{F}(y) \subset \Lambda$ (see Remark 3.3). Now s -minimality gives that $\bigcup_{i=1}^d f^i(\mathcal{F}^s(y))$ is a dense subset of Λ .

At this point, we concluded that every $z \in \Lambda$ is accumulated by an entire strong stable leaf $f^i(\mathcal{F}^s(y)) \subset \Lambda$, for some $i \in \{1, \dots, d\}$. Since the strong stable lamination is continuous and Λ is closed, we get that $\mathcal{F}^s(z) \subset \Lambda$, ending the proof of this Lemma. \square

We are now ready to prove of Theorem 3.2

Proof of Theorem 3.2. Observe that the interior of Λ , denoted by $\text{int}(\Lambda)$, is an invariant subset of Λ . Moreover, if Λ has non-empty interior, then it contains some strong stable disk. By Lemma 3.5, the set Λ contains the strong stable leaf of every point in Λ .

Suppose that the boundary $\partial\Lambda$ of Λ is non-empty. Let $z \in \partial\Lambda$ and consider the disk $D = \mathcal{F}_r^s(z) \subset \Lambda$. By Lemma 3.4, there is $N \in \mathbb{N}$ such that $f^{-N}(D)$

intersects $\text{int}(\Lambda)$. The f -invariance of $\text{int}(\Lambda)$ implies that $D \cap \text{int}(\Lambda) \neq \emptyset$. Now, choose some point x in this intersection and an open neighborhood B of x with $B \subset \text{int}(\Lambda)$. For each point $y \in B$ we consider its entire strong stable leave $\mathcal{F}^s(y)$, that is contained in Λ (recall Lemma 3.5). By the continuity of the strong stable foliation, the set $V = \bigcup_{y \in B} \mathcal{F}^s(y) \subset \Lambda$ is a neighborhood of $\mathcal{F}^s(x) = \mathcal{F}^s(z)$. Thus V is a neighborhood of z that is contained in Λ , contradicting the fact that $z \in \partial\Lambda$. Therefore $\partial\Lambda = \emptyset$, and consequently $\Lambda = M$. \square

We end this section by providing two technical results that will be necessary in Section 4.

First, let us recall that, by item (4) of Proposition 2.4, the partially hyperbolic splitting of a generic partially hyperbolic homoclinic class Λ extends to a neighborhood U of Λ in an invariant way. In addition, Lemma 5.3 and Remark 5.5 in [16] assure that the strong stable leave of any point in Λ that approximate a hyperbolic periodic point in Λ of index d^s (the dimension of the stable bundle) must transversally intersect the unstable manifold of this point. This is an important fact we are assuming during the proof of the following Lemma.

Lemma 3.6. *Let $f \in \mathcal{R}$ and $\Lambda_f(U) = H(p, f)$ be an isolated s -minimal partially hyperbolic homoclinic class of a hyperbolic periodic point p of index d^s . Then, the unstable manifold of p meets transversely any strong stable disk centered at a point in $\Lambda_f(U)$.*

Proof. Fix $x \in \Lambda_f(U)$, $r > 0$ and $\delta > 0$. Given $\varepsilon > 0$, Lemma 3.4 gives $N \in \mathbb{N}$ such that $f^{-N}(\mathcal{F}_r^s(x))$ contain a point y that is $\varepsilon/2$ -close to p . By taking ε sufficiently small, the disk $\mathcal{F}_\delta^s(y)$ intersect transversely $W_\varepsilon^u(\mathcal{O}_f(p))$. Moreover, by item *ii*) of Remark 3.3, N can be chosen big enough so that, as f contracts the stable leaves, $f^N(\mathcal{F}_\delta^s(y)) \subset \mathcal{F}_{2r}^s(x)$. This shows that $\mathcal{F}_{2r}^s(x)$ intersect transversely $W^u(\mathcal{O}_f(p))$. By the arbitrary choice of $x \in \Lambda_f(U)$ and $r > 0$, the conclusion follows. \square

Lemma 3.7. *Let $f \in \mathcal{R}$ and $\Lambda = H(p, f)$ be an isolated s -minimal partially hyperbolic set of some hyperbolic periodic point p of index d^s . Then, for every $x, y \in \Lambda$ satisfying $\mathcal{F}^s(x) \subset \overline{\mathcal{F}^s(y)}$ it holds that $\mathcal{F}_\Lambda^s(x) \subset \overline{\mathcal{F}_\Lambda^s(y)}$.*

Proof. Let $z \in \mathcal{F}_\Lambda^s(x)$, $r > 0$ and consider the disk $\mathcal{F}_r^s(z)$. By Lemma 3.6, $W^u(p)$ meets transversely $\mathcal{F}_r^s(z)$, say at the point w . Since $\mathcal{F}^s(x) \subset \overline{\mathcal{F}^s(y)}$, we also have an intersection point \hat{w} of $\mathcal{F}^s(y)$ and $W^u(p)$ that can be chosen arbitrarily close to w . From s -minimality, the orbit of $\mathcal{F}^s(p)$ accumulates at $\mathcal{F}^s(y)$ and thus intersect transversely $W^u(p)$ in a sequence of points that accumulate to \hat{w} . This sequence of points consist of transverse homoclinic points of p , so $\hat{w} \in \Lambda$. As r can be chosen arbitrarily small and \hat{w} can be chosen arbitrarily close to w , we conclude that $z \in \overline{\mathcal{F}_\Lambda^s(y)}$. Since it holds for every $z \in \mathcal{F}_\Lambda^s(x)$ we finally obtain that $\mathcal{F}_\Lambda^s(x) \subset \overline{\mathcal{F}_\Lambda^s(y)}$. \square

3.2. s -minimal attractors.

In what follows we study s -minimal attractors apart, with no similar statements to the case of u -minimal attractors ⁴.

The main result presented here is Theorem C. Before proving it, we need some intermediate results that also hold for $d^c \geq 1$.

In the next statements, the notation $\text{Per}_\sigma(f|_\Lambda)$ stands for the set of hyperbolic periodic points in Λ of index σ .

Lemma 3.8. *Let $\Lambda = \Lambda_f(U)$ be a partially hyperbolic attractor that is s -minimal, contains some strong stable disk, and has a point $p \in \text{Per}_{\text{ds}}(f|_\Lambda)$. Then Λ is the whole manifold.*

Proof. By Theorem 3.2, it suffices to prove that Λ has non-empty interior. Consider the periodic point $p \in \text{Per}_{\text{ds}}(f|_\Lambda)$. Then, for a small $\varepsilon > 0$, its local unstable manifold $W_\varepsilon^u(p)$ is a $(d^u + d^c)$ -dimensional embedded manifold contained in the attractor. By Lemma 3.5, the strong stable leaf of any point in Λ is contained in Λ . Thus the saturation of $W_\varepsilon^u(p)$ by its strong stable leaves contains an open subset of Λ , so Λ has non-empty interior. \square

The following proposition is a simplified version of Corollary B in [4] for the case of partially hyperbolic attractors.

Proposition 3.9 ([4]). *Fix $\alpha > 0$ and $f \in \text{Diff}^{1+\alpha}(M)$. If Λ is a partially hyperbolic set of f with $m(\Lambda) > 0$, then Λ contain some strong stable disk and some strong unstable disk.*

Lemma 3.10. *Let $f \in \text{Diff}^{1+\alpha}(M)$ and $\Lambda = \Lambda_f(U)$ be partially hyperbolic attractor that is s -minimal. If $\text{Per}_{\text{ds}}(f|_\Lambda) \neq \emptyset$ and $m(\Lambda) > 0$, then Λ is the whole manifold.*

Proof. By Proposition 3.9 there is a strong stable disk D contained in Λ . Now Lemma 3.8 implies the statement. \square

We are now ready to prove Theorem C.

Proof of theorem C. Since f is C^1 -generic and $\Lambda_f(U)$ is s -minimal, we can assume that $\Lambda_f(U)$ is generically s -minimal (see Proposition 2.5). Let \mathcal{U} be a compatible neighbourhood of f and \mathcal{J}_0 be the residual subset of \mathcal{U} of diffeomorphisms g such that $\Lambda_g(U)$ is s -minimal.

Claim 3.11. *For every $g \in \mathcal{J}_0$, $\varepsilon > 0$, and every hyperbolic periodic point $a \in \Lambda_g(U) \cap \text{Per}_{d^s+1}(g)$ it holds that*

$$\text{int}(W_\varepsilon^s(a) \cap \Lambda_g(U)) = \emptyset.$$

Here the interior refers to the topology of $W_\varepsilon^s(a)$.

⁴ Recall that by taking f^{-1} , the attractor becomes a repeller.

Proof of the claim. The proof is by contradiction. Assume that there are $\varepsilon > 0$ and $a \in \Lambda_g(U) \cap \text{Per}_{d^s+1}(g)$ such that $\text{int}(W_\varepsilon^s(a, g) \cap \Lambda_g(U))$ contains an open ball B of $W_\varepsilon^s(a, g)$. By saturating B with strong unstable leaves (which are subsets of the attractor $\Lambda_g(U)$) we get an open set (relative to the ambient manifold M) contained in $\Lambda_g(U)$. Thus $\Lambda_g(U)$ has non-empty interior and, by Theorem 3.2 it is the whole manifold, contradicting the fact that $\Lambda_g(U)$ is a proper attractor. \square

Consider a diffeomorphism f as in the statement of Theorem C and a pair of hyperbolic periodic points $p, q \in \Lambda_f(U)$ with indices d^s and $d^s + 1$, respectively (these points exist by item (2) of Proposition 2.4 and Remark 2.2). Let \mathcal{W}_p and $\mathcal{V}_{p,q}$ be the open sets given by items (3) and (5) of Proposition 2.4, respectively. By shrinking \mathcal{W}_p if necessary, we can assume that $\mathcal{W}_p \subset \mathcal{V}_{p,q}$, so the continuation q_g of q is well defined for every $g \in \mathcal{W}_p$.

Claim 3.12. *The map ϕ given by $g \mapsto W_\varepsilon^s(q_g, g) \cap \Lambda_g(U)$, defined on \mathcal{W}_p , is upper semicontinuous.*

Proof. Observe that, for every $g \in \mathcal{W}_p$, the set $\{\mathcal{F}_\varepsilon^u(x) \mid x \in W_\varepsilon^s(q_g, g) \cap \Lambda_g(U)\}$ is an open subset of $\Lambda_g(U)$. Since $W_\varepsilon^s(p_g, g)$ varies continuously, this observation shows that an upper discontinuity of ϕ would imply an upper discontinuity of $\Lambda_g(U)$. However, such a discontinuity for $\Lambda_g(U)$ is not possible as attractors vary upper semicontinuously. \square

As a consequence of this claim, there is a residual subset $\mathcal{J}_1 \subset \mathcal{W}_p$ consisting of continuity points of the map ϕ .

By Claim 3.11 and the definition of \mathcal{J}_1 we conclude that, for every $h \in \mathcal{J}_0 \cap \mathcal{J}_1$ (that is a subset of \mathcal{W}_p), there is a neighborhood \mathcal{U}_h of h such that

$$(3.1) \quad W_\varepsilon^s(q_g, g) \not\subset \Lambda_g(U) \quad \text{for all } g \in \mathcal{U}_h.$$

The set $\mathcal{V}_p = \bigcup_{h \in \mathcal{J}_0 \cap \mathcal{J}_1} \mathcal{U}_h$ is an open and dense subset of \mathcal{W}_p .

Claim 3.13. *For every $g \in \mathcal{V}_p$ the attractor $\Lambda_g(U)$ does not contain any strong stable disk, and consequently it has empty interior.*

Proof. Suppose that there is $g \in \mathcal{V}_p$ for which $\Lambda_g(U)$ has a strong stable disk $D \subset \Lambda_g(U)$. By the invariance and closeness of $\Lambda_g(U)$, any accumulation point of the backward orbit of D belongs to $\Lambda_g(U)$. By item (4) of Proposition 2.4, the closure of the negative orbit of D contains $H(p_g, g)$, so we conclude that $\overline{\mathcal{F}^s(p_g, g)} \subset \Lambda_g(U)$. Now, item (3) of Proposition 2.4 implies that $W^s(q_g, g) \subset \Lambda_g(U)$, contradicting Equation (3.1). \square

Recall that \mathcal{V}_p depends on the choice of $f \in \text{Diff}^1(M)$ and, since $f \in \overline{\mathcal{W}_p}$, we also have $f \in \overline{\mathcal{V}_p}$. Hence, to obtain item (1) of Theorem C, we apply Claim 3.13 with respect to every diffeomorphism in $\mathcal{R} \cap \mathcal{U}$. The union of all open sets obtained in this way is the announced open and dense subset \mathcal{V} of \mathcal{U} .

Fix $\alpha > 0$. To prove the second part of the theorem, observe that, if $g \in \mathcal{V} \cap \text{Diff}^{1+\alpha}(M)$ is such that $m(\Lambda_g(U)) > 0$, then it contains a strong

stable disk (see Proposition 3.9). This contradicts Claim 3.13, since we have taken $g \in \mathcal{V}$. This proves that the subset of \mathcal{U} for which $\Lambda_g(U)$ has zero Lebesgue measure contains every $C^{1+\alpha}$ diffeomorphism of \mathcal{V} .

In particular, for every C^2 diffeomorphisms g in \mathcal{V} , the attractor $\Lambda_g(U)$ has zero Lebesgue measure. Since the subset of C^2 diffeomorphisms in \mathcal{V} is C^1 -dense in \mathcal{V} , Corollary 2.7 implies that there is a residual (with respect to the C^1 topology) subset of \mathcal{V} where the attractors have zero Lebesgue measure. \square

4. SPECTRAL DECOMPOSITION

In this section we see how u - and s -minimal homoclinic classes are decomposed into a finite number of compact sets which are permuted by the dynamics and verify the strong recurrence property of mixing. Moreover, the number of pieces in this decomposition is exactly the minimal constant d in Definition 3.1. Let us describe it more precisely.

Definition 4.1 (Spectral decomposition). We say that a transitive compact invariant set Λ admits a *spectral decomposition* if there exist compact sets $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ satisfying:

- (1) $\Lambda = \bigcup_{i=1}^k \Lambda_i$.
- (2) There is a cyclic permutation $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, k\}$ such that $f(\Lambda_i) = \Lambda_{\sigma(i)}$ for all $i \in \{1, \dots, k\}$. In particular, Λ_i is periodic with period k .
- (3) They are pairwise disjoint: $\Lambda_i \cap \Lambda_j = \emptyset$ for all $i \neq j$ in $\{1, \dots, k\}$.
- (4) For every $i \in \{1, \dots, k\}$, Λ_i is topologically mixing for the map f^k .

We call the sets Λ_i the *basic components* or the *basic pieces* of Λ .

Remark 4.2. As the permutation in item (2) is cyclic, the period of any periodic point in Λ is a multiple of the number k of components of Λ .

The main results in this section are Theorem D and its robust version for robustly transitive attractors in Theorem E. All the statements and proves in this section deal only with the s -minimal case. The u -minimal case readily follows by applying these results to the inverse map f^{-1} .

To prove these theorems we start with some auxiliary lemmas.

Lemma 4.3. *Let $\Lambda = H(p, f)$ be an isolated s -minimal set with minimal constant d and $\text{index}(p) = d^s$. Let $x \in \Lambda$ and $k > 1$ be such that*

$$\bigcup_{i=1}^k \overline{\mathcal{F}_\Lambda^s(f^i(x))} = \Lambda.$$

Then $k \geq d$.

Proof. Fix $y \in \Lambda$. From s -minimality, we get that

$$\bigcup_{i=1}^d \overline{\mathcal{F}_\Lambda^s(f^i(y))} = \Lambda.$$

Then there is some $m \in \{1, \dots, d\}$ such that $x \in \overline{\mathcal{F}_\Lambda^s(f^m(y))}$. It follows from the continuity of the foliation that $\mathcal{F}^s(x) \subset \overline{\mathcal{F}^s(f^m(y))}$ (see Proposition 5.4 of [16]). By Lemma 3.7, we get that:

$$\Lambda = \bigcup_{i=1}^k \overline{\mathcal{F}_\Lambda^s(f^i(x))} \subset \bigcup_{i=1}^k \overline{\mathcal{F}_\Lambda^s(f^{m+i}(y))} = f^m \left(\bigcup_{i=1}^k \overline{\mathcal{F}_\Lambda^s(f^i(y))} \right) \subset \Lambda.$$

Thus $f^m(\bigcup_{i=1}^k \overline{\mathcal{F}_\Lambda^s(f^i(y))}) = \Lambda$, and consequently $\bigcup_{i=1}^k \overline{\mathcal{F}_\Lambda^s(f^i(y))} = \Lambda$. As it holds for every $y \in \Lambda$, the constant k satisfies the s -minimality condition. Now, the definition of minimal constant implies that $k \geq d$. \square

Lemma 4.4. *Let Λ be as in Lemma 4.3. For every $x \in \Lambda$ the sequence of sets $\{\overline{\mathcal{F}_\Lambda^s(f^n(x))}\}_{n=1}^d$ is pairwise disjoint.*

Proof. The proof is by contradiction. Suppose there is $z \in \overline{\mathcal{F}_\Lambda^s(f^i(x))} \cap \overline{\mathcal{F}_\Lambda^s(f^j(x))}$ for some $i < j$ in $\{1, \dots, d\}$. By Lemma 3.7, the set $\mathcal{F}_\Lambda^s(z)$ is contained in this intersection. Since $\mathcal{F}_\Lambda^s(z) \subset \overline{\mathcal{F}_\Lambda^s(f^j(x))}$, we obtain

$$(4.1) \quad \bigcup_{n=1}^{j-i} \overline{\mathcal{F}_\Lambda^s(f^n(z))} \subset \bigcup_{n=j+1}^{2j-i} \overline{\mathcal{F}_\Lambda^s(f^n(x))}.$$

Similarly, since $\mathcal{F}_\Lambda^s(z) \subset \overline{\mathcal{F}_\Lambda^s(f^i(x))}$, we have that $\mathcal{F}_\Lambda^s(f^{j-i}(z)) \subset \overline{\mathcal{F}_\Lambda^s(f^j(x))}$, and consequently we obtain

$$(4.2) \quad \bigcup_{n=j-i+1}^d \overline{\mathcal{F}_\Lambda^s(f^n(z))} \subset \bigcup_{n=j+1}^{d+i} \overline{\mathcal{F}_\Lambda^s(f^n(x))}.$$

Denoting $r = \max\{2j-i, d+i\}$, $w = f^j(x)$, and putting together Equations (4.1) and (4.2), we conclude that

$$\Lambda = \bigcup_{n=1}^d \overline{\mathcal{F}_\Lambda^s(f^n(z))} \subset \bigcup_{n=j+1}^r \overline{\mathcal{F}_\Lambda^s(f^n(x))} = \bigcup_{n=1}^{r-j} \overline{\mathcal{F}_\Lambda^s(f^n(w))}.$$

This contradicts Lemma 4.3, since $r-j = \max\{j-i, d-j+i\} < d$. \square

Now we are ready to prove Theorem D.

Proof of Theorem D. We have to prove items (1),(2),(3) and (4) of Definition 4.1 with $k = d$.

Take some $x \in \Lambda$ and set $\Lambda_i = f^i(\overline{\mathcal{F}_\Lambda^s(x)})$ for $i \in \{1, \dots, d\}$. Item (1) of Definition 4.1 is an immediate consequence of s -minimality.

For item (2), set $\sigma(i) = i + 1$ for $1 \leq i < d$ and $\sigma(d) = 1$. It is clear that $f(\Lambda_i) = \Lambda_{i+1} = \Lambda_{\sigma(i)}$ for all $1 \leq i < d$. So we only have to prove that $f(\Lambda_d) = \Lambda_{\sigma(d)} = \Lambda_1$.

Applying Lemma 4.4 to x and $f(x)$, and the using the fact that Λ is s -minimal, we have

$$\Lambda = \bigcup_{n=1}^d \overline{\mathcal{F}_\Lambda^s(f^n(x))} = \bigcup_{n=2}^{d+1} \overline{\mathcal{F}_\Lambda^s(f^n(x))},$$

where both unions consist of pairwise disjoint sets. Hence, "subtracting" $\bigcup_{n=2}^d \overline{\mathcal{F}_\Lambda^s(f^n(x))}$ in this equation, we obtain that $\overline{\mathcal{F}_\Lambda^s(f(x))} = \overline{\mathcal{F}_\Lambda^s(f^{d+1}(x))}$, which means that $\Lambda_1 = f(\Lambda_d)$.

Item (3) is just Lemma 4.4.

For item (4), fix $i \in \{1, \dots, d\}$ and two relative open sets A, B of Λ_i . Consider a hyperbolic periodic point $q \in A$ and $r > 0$ such that $\mathcal{F}_r^s(q) \cap \Lambda_i \subset A$. Let $\varepsilon > 0$ be such that every ε -dense subset of Λ_i intersects B .

From s -minimality, there is $k \in \mathbb{N}$ sufficiently big so that $f^{-k-n \cdot d}(\mathcal{F}_r^s(q))$ is ε -dense in Λ_i for every $n \in \mathbb{N}$. Clearly, k must be a multiple of d , as both $\mathcal{F}_r^s(q)$ and B belong to the same component Λ_i . Then, for some fixed $L \in \mathbb{N}$, we can write

$$f^{-d \cdot (L+n)}(\mathcal{F}_r^s(q)) \cap B \neq \emptyset, \text{ for every } n \in \mathbb{N}.$$

In particular, $f^{n \cdot d}(B) \cap A \neq \emptyset$ for every $n > L$. Since we have chosen A and B as arbitrary relative open subsets of Λ_i , we conclude that f^d is mixing on Λ_i . \square

Theorem 4.5. *Let $\Lambda = H(p, f)$ be as in Lemma 4.3 for some generic $f \in \mathcal{R}$. Then there is a neighborhood \mathcal{U} of f such that, for every $g \in \mathcal{U}$ that is s -minimal, the minimal constant of g is also d .*

Proof. Let m be the period of the hyperbolic periodic point p . By Theorem D and Remark 4.2, there is $n \in \mathbb{N}$ such that $m = n \cdot d$. From s -minimality, we get that $\Lambda = \bigcup_{n=1}^d \overline{\mathcal{F}_\Lambda^s(f^n(p))}$. By item (2) in definition 4.1, with $k = d$, for every $i \in \{1, \dots, d\}$ it holds that

$$(4.3) \quad \Lambda_i = \overline{\mathcal{F}_\Lambda^s(f^i(p))} = \overline{\mathcal{F}_\Lambda^s(f^{d+i}(p))} = \dots = \overline{\mathcal{F}_\Lambda^s(f^{(n-1)d+i}(p))}.$$

This equation implies that $\mathcal{F}^s(f^i(p))$ intersects transversally the unstable manifold of $f^{d+i}(p)$, $f^{2d+i}(p)$, \dots , and $f^{(n-1)d+i}(p)$. Clearly, these transverse intersections occur robustly in a small neighbourhood \mathcal{U} of f . Hence, by the λ -lemma, for every $g \in \mathcal{U}$ it holds that

$$(4.4) \quad \overline{\mathcal{F}_{\Lambda_g}^s(g^i(p))} = \overline{\mathcal{F}_{\Lambda_g}^s(g^{d+i}(p))} = \dots = \overline{\mathcal{F}_{\Lambda_g}^s(g^{(n-1)d+i}(p))}.$$

This shows that the number of pieces in the spectral decomposition of Λ_g for g in a small neighborhood of f cannot increase (is at most d).

On the other hand, the pairwise disjoint compact isolated sets $\{\Lambda_i\}_{i=1}^d$ admit upper semicontinuations for any diffeomorphism g sufficiently close to f , and the cyclic permutation given by f induces a cyclic permutation given by g on these continuations. Hence the number of components of $\Lambda_g(U)$ do not decrease in a small neighborhood of f .

As a conclusion, the spectral decomposition of g has exactly d components. Then d must be the minimal constant of the s -minimality of Λ_g . \square

Proof of Theorem E. By item (2) of Proposition 2.5, we can assume that f is either robustly s -minimal or robustly u -minimal. Without loss of generality, we admit that f is robustly s -minimal (with minimal constant d). We can also assume that $\Lambda_f(U)$ is robustly a homoclinic class, and that $\Lambda_g(U)$ vary continuously in a neighborhood of f (see Corollary 4.9 in [16]). Then $\Lambda_g(U)$ consist of d attractors of f^d that are the continuations of the components of $\Lambda_f(U)$. By theorem 4.5, the spectral decomposition of $\Lambda_g(U)$ has exactly d components, so they must coincide with the continuations of the pieces in the spectral decomposition of $\Lambda_f(U)$. \square

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